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The Theory of Degenerate Algebraical Curves and Surfaces.

By O. E. Glenn.

Chasles[*] arrived by geometrical intuition at the fundamentals of the theory of characteristics of plane curves.

An important principle at the basis of this theory is the fact that a system of curves defined by definite conditions will in general contain degenerate curves which, though degenerate, still satisfy all of the defining conditions of the system. It was this fact that led Cayley[†] to emphasize the importance of the general question of the degeneracy of curves as an object of investigation. This he did in an article "Sur les courbes aplaites" in volume 74 of the Comptes Rendus, and subsequently in a few notes to be found in his Works. The subject has nevertheless never been exhaustively investigated. A few important contributions have been published, and the question has been studied more or less incidentally in a number of places, but it would seem that much yet remains to be done.

As an application of some of his researches on symmetric functions, Junker[‡] studied in 1894 the decomposition of forms into linear factors. Hočevar[§] has treated the linear and quadratic factors of forms, and his work has been extended recently by O. Dorner.[||] Other references to work in this field are the following:

H. M. Taylor: "On the Decomposition of a Cubic Curve," *Proceedings London Mathematical Society*, Vol. XXVIII.

Meyer: "Zur Theorie der reduciblen ganzen Functionen von n Variablen," *Math. Annalen*, Bd. XXX.

Gordan: "Das Zerfallen der Curven in gerade Linien," *Math. Annalen*, Bd. XLV.

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[*] Chasles: *Comptes Rendus*, t. 64, pp. 799, 1079.

[†] Cayley: *Comptes Rendus*, t. 74, p. 708.

[‡] Junker: *Math. Annalen*, Bd. XLV, S. 1-84.

[§] Hočevar: *Wiener Sitzungsberichte*, 1904 and 1907.

[||] Dorner: "Ueber Teiler von Formen," *Progr. Königsberg*, 1908.

Brill: "Ueber die Zerfallung der Ternärformen in linearen Factoren," *Math. Annalen*, Bd. L.

Hadamard: "Sur les conditions de décomposition d'une forme ternaire," Bordeaux, *Proces-Verbaux*, 1897.

Most of the published articles treat the formal side of the question only. I have* in this paper sought to develop methods which would throw light on the geometrical phases of the theory, and to interpret various algebraical conditions which cause degeneracy. The analytical theory forming the first part of the paper is essentially a generalization of certain results of Junker in his memoir, previously quoted, on symmetric functions of several groups of variables.

Cayley introduced the name "penultimate curve" for the locus whose equation contains certain infinitesimal coefficients the vanishing of which renders the equation factorable into an ultimate form

$$P^{\alpha} Q^{\beta} R^{\gamma} \ldots\ldots = 0.$$

We shall also use the term "penultimate surface" for the locus of a degenerating quaternary form.

PART I.

THE ASYMPTOTIC FORMS OF A GIVEN FORM.

§ 1. *A Class of Symmetric Functions.*

Let the m roots of

$$f(x) = x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots\ldots + a_m = 0$$

be $x_1, x_2, \ldots\ldots, x_m$. Let the totality of the sums of the roots taken in pairs, and of the products of the roots in pairs be respectively

$$\xi_i, \ \eta_j \ [i, j = 1, 2, \ldots\ldots, \tfrac{1}{2} m (m-1)].$$

The elementary symmetric functions of the ξ_i and of the η_j can be evaluated in terms of the coefficients a of $f(x)$. For

$$\Sigma \xi_1 \xi_2 \ldots\ldots \xi_k = \phi_k = J_k \ (x_1, x_2, \ldots\ldots, x_m), \tag{1}$$

$$\Sigma \eta_1 \eta_2 \ldots\ldots \eta_k = \psi_k = I_k \ (x_1, x_2, \ldots\ldots, x_m) \quad (k = 1, 2, \ldots\ldots, m), \tag{2}$$

where I and J are symmetric in the x's.

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* The reviewer who may wish a summary of the principal results of the paper may read those parts which have been printed in italics.

In fact we have, using undetermined coefficients,

$$\phi_k = \Sigma\, \lambda_{abc\ldots\ldots}\, a_1^a a_2^b a_3^c \ldots\ldots \qquad (a + 2b + 3c + \ldots\ldots = k),$$
$$\phi_{k-1} = \Sigma\, \mu_{\alpha\beta\gamma\ldots\ldots}\, a_1^\alpha a_2^\beta a_3^\gamma \ldots\ldots \qquad (\alpha + 2\beta + 3\gamma + \ldots\ldots = k-1),$$

and when the μ coefficients are known, the coefficients λ and thus ϕ_k itself may be determined in the well-known manner by operating upon (1) as follows:

$$\sum \frac{\partial J_k}{\partial x_1} = m\frac{\partial \phi_k}{\partial a_1} + (m-1)\, a_1 \frac{\partial \phi_k}{\partial a_2} + (m-2)\, a_2 \frac{\partial \phi_k}{\partial a_3} + \ldots\ldots$$

This gives

$$\begin{aligned}(m^2 - m - 2k + 2)\, &\Sigma\, \mu_{\alpha\beta\gamma\ldots\ldots}\, a_1^\alpha a_2^\beta a_3^\gamma \ldots\ldots \\ &= m\,\Sigma\, a\, \lambda_{abc\ldots\ldots}\, a_1^{a-1} a_2^b a_3^c \ldots\ldots + (m-1)\,\Sigma\, b\, \lambda_{abc\ldots\ldots}\, a_1^{a+1} a_2^{b-1} a_3^c \ldots\ldots \\ &\qquad + (m-2)\,\Sigma\, c\, \lambda_{abc\ldots\ldots}\, a_1^a a_2^{b+1} a_3^{c-1} \ldots\ldots + \ldots\ldots,\end{aligned}$$

whence the recurring relation for the λ's

$$\begin{aligned}(m^2 - m - 2k + 2)\, \mu_{\alpha\beta\gamma\ldots\ldots} &= m\,(\alpha + 1)\, \lambda_{\alpha+1\,\beta\,\gamma\ldots\ldots} + (m-1)\,(\beta+1)\, \lambda_{\alpha-1\,\beta+1\,\gamma\ldots\ldots} \\ &\qquad + (m-2)\,(\gamma + 1)\, \lambda_{\alpha\,\beta-1\,\gamma+1\ldots\ldots} + \ldots\ldots \qquad (3)\end{aligned}$$

The number of these relations (number of μ's) equals the number of sets of solutions in positive integers including zero of

$$\alpha + 2\beta + 3\gamma + \ldots\ldots = k - 1,$$

whereas the number of λ's equals the (greater) number of solutions of

$$a + 2b + 3c + \ldots\ldots = k.$$

Thus the equations (3) would be insufficient in number to determine the λ's, but the deficiency is always easily made up since some of the λ's can be found by inspection. Hence, starting with the known function

$$\phi_1 = (m-1)\, a_1,$$

we can determine by (3) the other ϕ_k in succession. For example,

$$\phi_2 = \tfrac{1}{2}(m-1)(m-2)\, a_1^2 + (m-2)\, a_2,$$

and the recurring formula (3) gives, for $k = 3$,

$$3m\,\lambda_{300} + (m-1)\,\lambda_{110} = \binom{m-1}{2}(m^2 - m - 4),$$
$$m\,\lambda_{110} + (m-2)\,\lambda_{001} = (m-2)(m^2 - m - 4).$$

Also

$$\lambda_{300} = \binom{m-1}{2}.$$

The solution of these three equations determines

$$\phi_3 = \lambda_{300}\, a_1^3 + \lambda_{110}\, a_1 a_2 + \lambda_{001}\, a_3$$
$$= \binom{m-1}{3} a_1^3 + (m-2)^2 a_1 a_2 + (m-4)\, a_3 .$$

Similarly, $\phi_6 = a_1 a_2 a_3 - a_1^2 a_4 - a_3^2\ (m < 5)$, etc.

The functions ψ_k may be determined in the same manner or, within the limitations of the computed tables of symmetric functions,* these may be advantageously employed. Thus $\psi_5 = a_2 a_4^2\ (m < 5)$, $\psi_{\frac{1}{2}m(m-1)} = a_m^{m-1}$, etc.

A table of the functions ϕ and ψ follows:

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
$(m-1)\, a_1$	$\binom{m-1}{2} a_1^2$	$\binom{m-1}{3} a_1^3$	$\binom{m-1}{4} a_1^4$	$\binom{m-1}{5} a_1^5$
	$+(m-2)\, a_2$	$+(m-2)^2 a_1 a_2$	$+(m-2)\binom{m-2}{2} a_1^2 a_2$	$+(m-2)\binom{m-2}{3} a_1^3 a_2$
		$+(m-4)\, a_3$	$+(m-3)^2 a_1 a_3$	$+\frac{1}{2}(m-3)(m^2-6m+10)\, a_1^2 a_3$
			$+\binom{m-2}{2} a_2^2$	$+\binom{m-2}{2}(m-3)\, a_1 a_2^2$
			$+(m-8)\, a_4$	$+(m^2-10m+20)\, a_1 a_4$
				$+(m-3)(m-4)\, a_2 a_3$
				$+(m-16)\, a_5$

ψ_1: a_2

ψ_2: $a_1 a_3 - a_4$

ψ_3: $a_1^2 a_4 - 2 a_2 a_4 + a_3^2 - a_1 a_5 + a_6$

ψ_4: $a_1 a_3 a_4 - a_4^2 - 3 a_1 a_2 a_5 + a_3 a_5 + a_1^3 a_5 - a_1^2 a_6 + 2 a_2 a_6 + 3 a_1 a_7 - a_8 .$

§ 2. *Nature of Functions* ϕ, ψ.

If the quadratic

$$x^2 - \xi x - \eta$$

divides $f(x)$, then ξ and η are roots respectively of

$$\left.\begin{aligned} \Phi_\xi^{(m)} &= \xi^{(m)} + \phi_1 \xi^{(m)-1} + \phi_2 \xi^{(m)-2} + \ldots\ldots + \phi_{(m)} \\ \Psi_\eta^{(m)} &= \eta^{(m)} + \psi_1 \eta^{(m)-1} + \psi_2 \eta^{(m)-2} + \ldots\ldots + \psi_{(m)} \quad [(m) = \tfrac{1}{2} m (m-1)]. \end{aligned}\right\} \quad (4)$$

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* Tables have been computed up to the weight 12 inclusive. See the Cayley-Hirsch tables in Salmon's Modern Higher Algebra for weights 1 to 10 inclusive, and Řehořovský: "Tafeln der sym. Functionen vom Gewichte 11 und 12," *Wiener Denkschriften*, 1882.

For illustration, if $f(x)$ has rational coefficients and is resolvable into quadratic factors in the absolute field, it can be resolved by factoring $\phi_{(m)}$ and a_m, trying the factors, as ξ, η, in $x^2 - \xi x - \eta$ and retaining those quadratics which are found to divide $f(x)$. Any equation can be tested for this resolvability by a finite number of trials.*

Generalization.

Let the totality of sums of the roots of $f(x)$ taken ν in a sum, of the sums of products two at a time of the roots of each of these sums, of the sums of products three at a time of the roots of each of these original sums, and so on, be respectively represented by

$$\xi_{1i},\ \xi_{2j},\ \xi_{3k},\ \ldots\ldots \quad [i, j, k, \ldots\ldots = 1, 2, \ldots\ldots, C^m_\nu].$$

Then we have

$$\begin{aligned}
&\Sigma\, \xi_{11}\xi_{12} \ldots\ldots \xi_{1h} = J_{1h}(x_1, x_2, \ldots\ldots, x_m) = \phi_{1h}(a),\\
&\Sigma\, \xi_{21}\xi_{22} \ldots\ldots \xi_{2h} = J_{2h}(x_1, x_2, \ldots\ldots, x_m) = \phi_{2h}(a),\\
&\Sigma\, \xi_{31}\xi_{32} \ldots\ldots \xi_{3h} = J_{3h}(x_1, x_2, \ldots\ldots, x_m) = \phi_{3h}(a) \quad [h = 1, 2, \ldots\ldots, C^m_\nu],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

where J_{gh} is symmetric in the x's, and hence $\phi_{gh}(a)$ rational in the a's. *In order that*

$$\alpha^\nu_x = x^\nu - \sum_{g=1}^{\nu} \xi_g\, x^{\nu-g}$$

may be a divisor of $f(x)$, ξ_g *must be a root of*

$$\Phi^{C^m_\nu}_{g\xi} = \xi^{C^m_\nu} + \phi_{g1}\,\xi^{C^m_\nu - 1} + \phi_{g2}\,\xi^{C^m_\nu - 2} + \ldots\ldots + \phi_{gC^m_\nu} \qquad (5)$$

and conversely. For example, when $m = 5$, $\nu = 4$, $h = 1$,

$$\begin{aligned}
\phi_{11} &= 4a_1,\\
\phi_{12} &= 6a_1^2 + a_2,\\
\phi_{13} &= 4a_1^3 + 3a_1a_2 - a_3,\\
\phi_{14} &= a_1^4 + 3a_1^2a_2 - 2a_1a_3 + a_4,\\
\phi_{15} &= a_1^3a_2 - a_1^2a_3 + a_1a_4 - a_5.
\end{aligned}$$

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* Let $f(x) = x^4 + 2x^2 + x + 2 = 0$. Then ξ divides $a_1a_2a_3 - a_1^2a_4 - a_3^2 = -1$; $\xi = \pm 1$, while η divides $a_4 = 2$; $\eta = \pm 1, \pm 2$. By trial we retain the pairs $(\xi, \eta) = (-1, -1), (1, -2)$; *i. e.*, $f(x) = (x^2 + x + 1)(x^2 - x + 2)$.

§ 3. *Quadratic Factors of Forms.*

The decomposition of forms into linear factors has been studied by Junker, who gave a method of determining the component forms, and discovered relations which exist among the coefficients of the form when it is degenerate. Let us consider forms which are decomposable into quadratic factors.

Let the ternary form

$$\begin{aligned} f_{3m}(x, y, 1) = a_{0m} y^m + a_{1m-1} y^{m-1} x + \ldots\ldots + a_{m0} x^m + a_{0m-1} y^{m-1} \\ + \ldots\ldots + a_{01} y + a_{10} x + a_{00} \end{aligned}$$

equal the product of the $\frac{1}{2}m$ quadratic forms

$$\sigma_i = s_i x^2 + t_i xy + u_i y^2 + v_i x + w_i y + r_i \qquad (i = 1, 2, \ldots\ldots, \tfrac{1}{2}m).$$

Then we have

$$\begin{aligned} &\Sigma s_1 s_2 \ldots\ldots s_g = a_{m0}, \\ &\Sigma s_1 s_2 \ldots\ldots s_{g-1} t_g = a_{m-11}, \\ &\Sigma s_1 s_2 \ldots\ldots s_{g-1} u_g + \Sigma s_1 s_2 \ldots\ldots s_{g-2} t_{g-1} t_g + \ldots\ldots = a_{m-22}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\Sigma u_1 u_2 \ldots\ldots u_g = a_{0m}, \\ &\Sigma s_1 s_2 \ldots\ldots s_{g-1} v_g = a_{m-10}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\Sigma u_1 u_2 \ldots\ldots u_{g-1} w_g = a_{0m-1} \qquad [g = \tfrac{1}{2}m], \end{aligned}$$

from which we get the following differential relations:

$$\sum \frac{\partial a_{m-kk}}{\partial s_1} v_1 + \sum \frac{\partial a_{m-kk}}{\partial t_1} w_1 = a_{m-k-1k} \qquad (k = 0, 1, 2, \ldots\ldots, \overline{m-1}). \tag{6}$$

Without affecting the generality of f_{3m}, we take $u_i = a_{0m} = 1$, and write

$$f_{3m} = y^m + \sum_{r=1}^{m} (a_{rm-r} x^r + a_{r-1\,m-r} x^{r-1} + \ldots\ldots + a_{0m-r}) y^{m-r},$$

$$\sigma_i = y^2 + (t_i x + w_i) y + (s_i x^2 + v_i x + r_i).$$

Hence by § 2

$$\Phi^{(m)}_{-(t_i x + w_i)} = 0, \quad \Psi^{(m)}_{-(s_i x^2 + v_i x + r_i)} = 0,$$

wherein the usual coefficients a_r have been replaced by

$$a_{rm-r} x^r + a_{r-1\,m-r} x^{r-1} + \ldots\ldots + a_{0m-r} \qquad (r = 1, 2, \ldots\ldots, m).$$

It follows that the coefficients t_i and s_i are respectively roots of the equations

$$\Phi^{(m)}_{-t} = t^{(m)} - \phi_1 t^{(m)-1} + \phi_2 t^{(m)-2} - \ldots\ldots + (-1)^{(m)} \phi_{(m)} = 0, \tag{7}$$

$$\Psi^{(m)}_{-s} = s^{(m)} - \psi_1 s^{(m)-1} + \psi_2 s^{(m)-2} - \ldots\ldots + (-1)^{(m)} \psi_{(m)} = 0, \tag{8}$$

in which the usual coefficients a_r (see § 2) are now replaced by a_{rm-r}. Moreover the coefficients ϕ, ψ are rational in the elementary symmetric functions a of the groups of quantities

$$s_i,\ t_i,\ u_i,\ v_i,\ w_i,\ r_i \qquad (i = 1, 2, \ldots, g).$$

When these symmetric functions a are expressed in terms of the $s_i, t_i, \ldots$, and s, t are replaced by s_j, t_j, respectively, for any value of j, Φ_{-t} and Ψ_{-s} vanish identically. When so expressed, we write them $I(s_i, \ldots)$, $J(s_i, \ldots)$ respectively. Hence

$$\Phi(a, t_j) = I(s_i, \ldots; t_j) \equiv 0,$$
$$\Psi(a, s_j) = J(s_i, \ldots; s_j) \equiv 0.$$

Now in I let s_i, t_i be replaced respectively by $s_i + \lambda v_i$, $t_i + \lambda w_i$ (t by $t + \lambda w$). Then expand by Taylor's theorem

$$I(s_i + \lambda v_i,\ t_i + \lambda w_i,\ \ldots;\ t + \lambda w),$$

and by equating coefficients we get

$$\sum \frac{\partial I}{\partial s_1} v_1 + \sum \frac{\partial I}{\partial t_1} w_1 + \frac{\partial I}{\partial t} w = \sum \frac{\partial \Phi}{\partial s_1} v_1 + \sum \frac{\partial \Phi}{\partial t_1} w_1 + \frac{\partial \Phi}{\partial t} w.$$

But

$$\sum \frac{\partial \Phi}{\partial s_1} v_1 + \sum \frac{\partial \Phi}{\partial t_1} w_1 = \sum \frac{\partial \Phi}{\partial a_{m-kk}} \Big(\sum \frac{\partial a_{m-kk}}{\partial s_1} v_1 + \sum \frac{\partial a_{m-kk}}{\partial t_1} w_1 \Big)$$
$$= \sum \frac{\partial \Phi}{\partial a_{m-kk}} a_{m-k-1k} \quad \text{(from (6))}.$$

Therefore we have for the determination of w_j ($j = 1, 2, \ldots, g$)

$$w_j \frac{\partial \Phi_{-t_j}}{\partial t_j} + \sum_k \frac{\partial \Phi_{-t_j}}{\partial a_{m-kk}} a_{m-k-1k} = 0. \tag{9}$$

By a similar expansion of $\psi(a, s)$ we get for the determination of v_j

$$v_j \frac{\partial \Psi_{-s_j}}{\partial s_j} + \sum_k \frac{\partial \Psi_{-s_j}}{\partial a_{m-kk}} a_{m-k-1k} = 0. \tag{10}$$

Thus the factors σ_i *of* f_{3m} *are completely determined from* (7), (8), (9), (10).

Next let the quaternary form

$$f_{4m}(x, y, z, 1) = y^m + \sum_{r=1}^{m} \Big[\sum_{s=0}^{r} (a_{s\,m-r\,r-s} x^s + a_{s-1\,m-r\,r-s} x^{s-1} + \cdots + a_{0\,m-r\,r-s}) z^{r-s} \Big] y^{m-r}$$

equal the product of the quadratic forms

$$\tau_i = y^2 + (o_i x + s_i z + u_i) y + (n_i x^2 + q_i x z + r_i z^2 + t_i x + v_i z + w_i)$$
$$[i = 1, 2, \ldots, g = \tfrac{1}{2} m].$$

Then we have, as in (6),

$$\left.\begin{aligned}
\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial n_1}t_1+\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial o_1}u_1+\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial q_1}v_1&=a_{s-1\,m-r\,r-s}\quad(s>0),\\
\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial o_1}t_1+\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial p_1}u_1+\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial s_1}v_1&=a_{s\,m-r-1\,r-s}\quad(r<m),\\
\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial q_1}t_1+\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial s_1}u_1+\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial r_1}v_1&=a_{s\,m-r\,r-s-1}\quad(s<r),
\end{aligned}\right\}\quad(11)$$

and also as in (7), (8) that the coefficients o_i and n_i are roots of the respective equations

$$\Phi_{-o}^{(m)}=o^{(m)}-\phi_1 o^{(m)-1}+\phi_2 o^{(m)-2}-\ldots\ldots+(-1)^{(m)}\phi_{(m)}=0,\qquad(12)$$

$$\Psi_{-n}^{(m)}=n^{(m)}-\psi_1 n^{(m)-1}+\psi_2 n^{(m)-2}-\ldots\ldots+(-1)^{(m)}\psi_{(m)}=0.\qquad(13)$$

In these the usual coefficients a_r [§ 2] are replaced by $a_{r\,m-r\,0}$ $[r=1, 2, \ldots, m]$. Now when the coefficients $a_{r\,m-r\,0}$ in these functions (12), (13) are expressed as elementary symmetric functions of the groups of variables

$$n_i,\ o_i,\ p_i,\ q_i,\ r_i,\ s_i,\ t_i,\ u_i,\ v_i,\ w_i\quad[i=1, 2, \ldots, g],$$

we write them as

$$I(n_i,\ o_i,\ \ldots,\ w_i;\ o),\quad J(n_i,\ o_i,\ \ldots,\ w_i;\ n)$$

respectively. From the expansion of

$$I(n_i+\lambda t_i,\ o_i+\lambda u_i,\ p_i,\ q_i+\lambda v_i,\ r_i,\ \ldots,\ w_i;\ o+\lambda u),$$

we get

$$\begin{aligned}
&\sum\frac{\partial I}{\partial n_1}t_1+\sum\frac{\partial I}{\partial o_1}u_1+\sum\frac{\partial I}{\partial q_1}v_1+\frac{\partial I}{\partial q}v\\
&\quad=\sum\frac{\partial\Phi}{\partial a_{m-k\,k\,0}}\left[\sum\frac{\partial a_{m-k\,k\,0}}{\partial n_1}t_1+\sum\frac{\partial a_{m-k\,k\,0}}{\partial o_1}u_1+\sum\frac{\partial a_{m-k\,k\,0}}{\partial q_1}v_1\right]+\frac{\partial\Phi}{\partial o}u\\
&\quad=\sum\frac{\partial\Phi}{\partial a_{m-k\,k\,0}}a_{m-k-1\,k\,0}+\frac{\partial\Phi}{\partial o}u\qquad\text{(from (11))}.
\end{aligned}$$

Hence we have for the determination of u_j

$$u_j\frac{\partial\Phi_{-o_j}}{\partial o_j}+\sum_k\frac{\partial\Phi_{-o_j}}{\partial a_{m-k\,k\,0}}a_{m-k-1\,k\,0}=0.\qquad(14)$$

Similarly from the expansion of

$$J(n_i+\lambda t_i,\ o_i+\lambda u_i,\ p_i,\ q_i+\lambda v_i,\ r_i,\ \ldots,\ w_i;\ n+\lambda t)$$

and the first equation of (11) there results

$$t_j \frac{\partial \Psi_{-n_j}}{\partial n_j} + \sum_k \frac{\partial \Psi_{-n_j}}{\partial a_{m-k\,k\,0}} a_{m-k-1\,k\,0} = 0. \tag{15}$$

Again the coefficients r_i are roots of the equation

$$\Psi_{-r}^{(m)} = r^{(m)} - \psi_1 r^{(m)-1} + \psi_2 r^{(m)-2} - \ldots\ldots + (-1)^{(m)} \psi_{(m)} = 0, \tag{16}$$

wherein the usual a_r [§ 2] are replaced by $a_{0\,m-r\,r}$ $[r = 1, \ldots., m]$. Hence, from the third equation of (11) we have, by expanding

$$\Psi(a) = J(n_i,\ o_i,\ p_i,\ q_i + \lambda t_i,\ r_i + \lambda v_i,\ s_i + \lambda u_i,\ t_i,\ \ldots.,\ w_i;\ r + \lambda v),$$

$$v_j \frac{\partial \Psi_{-r_j}}{\partial r_j} + \sum_k \frac{\partial \Psi_{-r_j}}{\partial a_{0\,m-k\,k}} a_{0\,m-k\,k-1} = 0, \tag{17}$$

a formula for the determination of v_j.

Next let us write the form f_{4m} as a quantic in the three variables x, y, w, taking $z = 1$. That is, let

$$f_{4m}(x, y, 1, w) = \prod_{i=1}^{g} \tau_i(x, y, 1, w). \tag{18}$$

Then we have

$$f_{4m}(x, y, 1, w) = y^m + \sum_{r=1}^{m} \Big[\sum_{s=0}^{r} (a_{s\,m-r\,0}\, x^s + a_{s-1\,m-r\,1}\, x^{s-1} + \ldots. + a_{0\,m-r\,s})\, w^{r-s}\Big]\, y^{m-r},$$

$$\tau_i = y^2 + (o_i x + u_i w + s_i) y + (n_i x^2 + t_i x w + w_i w^2 + q_i x + v_i w + r_i)$$
$$(i = 1, \ldots., g).$$

It follows that the u_j determined by (14) satisfy the equation

$$\Phi_{-u}^{(m)} = u^{(m)} - \phi_1 u^{(m)-1} + \phi_2 u^{(m)-2} - \ldots. + (-1)^{(m)} \phi_{(m)} = 0, \tag{19}$$

wherein the usual a_r [§ 2] are replaced by $a_{0\,m-r\,0}$ $[r = 1, \ldots., m]$, and that we again get equation (13) for the determination of the n_j. Also, on account of the relation (18), we have, as in (6) and (11),

$$\Big(\sum \frac{\partial}{\partial t_1} q_1 + \sum \frac{\partial}{\partial u_1} s_1 + \sum \frac{\partial}{\partial w_1} v_1\Big) a_{s\,m-r\,0} = a_{s\,m-r\,1} \quad \begin{bmatrix} s = 0, 1, \ldots., r \\ r = 1, 2, \ldots., m \end{bmatrix},$$

$$\Big(\sum \frac{\partial}{\partial n_1} q_1 + \sum \frac{\partial}{\partial o_1} s_1 + \sum \frac{\partial}{\partial t_1} v_1\Big) a_{s\,m-r\,0} = a_{s-1\,m-r1} \quad (s > 0).$$

Hence, by applying Taylor's expansion as above we obtain for the determination of s_j, q_j respectively

$$s_j \frac{\partial \Phi_{-u_j}}{\partial u_j} + \sum_k \frac{\partial \Phi_{-u_j}}{\partial a_{0\,m-k\,0}} a_{0\,m-k\,1} = 0, \tag{20}$$

$$q_j \frac{\partial \Psi_{-n_j}}{\partial n_j} + \sum_k \frac{\partial \Psi_{-n_j}}{\partial a_{k\,m-k\,0}} a_{k-1\,m-k\,1} = 0. \tag{21}$$

The coefficient w_j satisfies an equation analogous to (13). *Hence equations* (12), (13), (14), (15), (16), (17), (20), (21) *completely determine the component factors* τ *of* f_{4m}.

§ 5. *Cubic Factors.*

If the ternary form

$$f_{3m} = a_{0m}\, y^m + \sum_{r=1}^{m} (a_{r\,m-r}\, x^r + a_{r-1\,m-r}\, x^{r-1} + \ldots . + a_{0\,m-r})\, y^{m-r}$$

can be split into $g = \frac{1}{3} m$ cubic factors

$$\rho_i = q_i y^3 + (p_i x + t_i) y^2 + (o_i x^2 + s_i x + v_i) y + (n_i x^3 + r_i x^2 + u_i x + w_i) \qquad (i = 1, \ldots, g),$$

its coefficients are connected with the coefficients of these cubics by the relations

$$\left.\begin{aligned} \Big(\sum \frac{\partial}{\partial n_1} r_1 + \sum \frac{\partial}{\partial o_1} s_1 + \sum \frac{\partial}{\partial p_1} t_1\Big)\, a_{m-r\,r} &= a_{m-r-1r}, \\ \Big(\sum \frac{\partial}{\partial o_1} r_1 + \sum \frac{\partial}{\partial p_1} s_1 + \sum \frac{\partial}{\partial q_1} t_1\Big)\, a_{m-r\,r} &= a_{m-r\,r-1}, \\ \Big(\sum \frac{\partial}{\partial w_i} u_i + \sum \frac{\partial}{\partial v_i} s_i + \sum \frac{\partial}{\partial t_i} p_i\Big)\, a_{0\,m-r} &= a_{1\,m-r}. \end{aligned}\right\} \tag{22}$$

Also the coefficients p, o, n satisfy the following equations respectively:

$$\Phi^{C^m_3}_{1,-p} = p^{C^m_3} - \phi_{11}\, p^{C^m_3-1} + \phi_{12}\, p^{C^m_3-2} - \ldots . + (-1)^{C^m_3} \phi_{1\,C^m_3} = 0, \tag{23}$$

$$\Phi^{C}_{2,-o} = o^{C^m_3} - \phi_{21}\, o^{C^m_3-1} + \phi_{22}\, o^{C^m_3-2} - \ldots . + (-1)^{C^m_3} \phi_{2\,C^m_3} = 0, \tag{24}$$

$$\Phi^{C^m_3}_{3,-n} = n^{C^m_3} - \phi_{31}\, n^{C^m_3-1} + \phi_{32}\, n^{C^m_3-2} - \ldots . + (-1)^{C^m_3} \phi_{3\,C^m_3} = 0, \tag{25}$$

wherein the coefficient a_r of § 2 are now replaced by $a_{r\,m-r}$ $[r = 1, 2, \ldots, m]$. Hence by Taylor's theorem and the first equation of (22) we get

$$t_j \frac{\partial \Phi_{1,-p_j}}{\partial p_j} + \sum_k \frac{\partial \Phi_{1,-p_j}}{\partial a_{m-k\,k}}\, a_{m-k-1\,k} = 0, \tag{26}$$

$$s_j \frac{\partial \Phi_{2,-o_j}}{\partial o_j} + \sum_k \frac{\partial \Phi_{2,-o_j}}{\partial a_{m-k\,k}}\, a_{m-k-1\,k} = 0, \tag{27}$$

$$r_j \frac{\partial \Phi_{3,-n_j}}{\partial n_j} + \sum_k \frac{\partial \Phi_{3,-n_j}}{\partial a_{m-k\,k}}\, a_{m-k-1\,k} = 0. \tag{28}$$

Again writing f_{3m} as a homogeneous form in x, y, z and then taking $x = 1$, we see that v_i and w_i satisfy the respective equations

$$\Phi^{C^m_3}_{2,-v} = 0, \quad \Phi^{C^m_3}_{3,-w} = 0, \tag{29}$$

wherein the a_r of § 2 are replaced by $a_{0\,m-r}$ $[r = 1, 2, \ldots, m]$. Hence by Taylor's theorem and the third equation of (22) we get for the determination of the remaining coefficients u_j

$$u_j \frac{\partial \Phi_{3,\,-w_j}}{\partial w_j} + \sum_k \frac{\partial \Phi_{3,\,-w_j}}{\partial a_{0\,m-k}} a_{1\,m-k} = 0. \tag{30}$$

The cubic factors of f_{3m} are completely determined by formulas (23)....(30).

§ 6. *Asymptotic Forms of a Given Form.*

Using a well-known notation let us write

$$f_{3m}(x, y, z) = u_m + u_{m-1} z + u_{m-2} z^2 + \ldots + u_0 z^m = \sum_{i=0}^{m} u_{m-i} z^i.$$

Let the expressions

$$v_{1\tau_1},\ v_{2\tau_2},\ \ldots,\ v_{\nu\tau_\nu} \quad (\Sigma \tau_i = m)$$

be any set of real factors of u_m which are algebraically prime to each other. Let u_{m-1}/u_m be resolved into partial fractions, and suppose that the numerators corresponding respectively to the above denominators are

$$v_{1\,\tau_1-1},\ v_{2\,\tau_2-1},\ \ldots,\ v_{\nu\,\tau_\nu-1}.$$

Then

$$f_{3m} = \zeta_{1\tau_1} \zeta_{2\tau_2} \ldots \zeta_{\nu\tau_\nu} + z^2 \mu_{m-2} = \prod_i \zeta_{i\tau_i} + z^2 \mu_{m-2}, \tag{31}$$

where

$$\zeta_{i\tau_i} = v_{i\tau_i} + v_{i\,\tau_i-1} z + R_{i\,\tau_i-2} z^2,$$

the $(\tau_i - 2)$-ic R being arbitrary. The forms $\zeta_{i\tau_i}$ will be called asymptotic forms of f_{3m}. The $(m - 2)$-ic μ_{m-2} will be called the satellite form of f_{3m}. Both μ_{m-2} and the satellite forms ζ are affected as to form by the choice of $R_{i\,\tau_i-2}$ $(i = 1, 2, \ldots, \nu)$.

Now if the $v_{i\tau_i}$ are the real factors of u_m of lowest degree prime to each other, assuming for the present that there are no repeated quadratic factors and no linear factor of multiplicity exceeding two, we have $\tau_i = 1$ or 2. We first show that *if by small variations of its coefficients a form f_{3m} decomposes into a form of degree $(m - k)$ and a factor of degree k $(= 1, 2)$, then this factor is identical with one of f_{3m}'s asymptotic forms ζ_{i_k}.*

Take $R_{i\,\tau_i-2} = 0$, and place $z = 1$ in (31). Then we have

$$\prod_{i=1}^{\nu} \zeta_{i\tau_i} = u_m + u_{m-1} + g_{m-2} = h_{3m}(x, y, 1).$$

This form h_{3m} is thus degenerate and formulas (7), (8), (9), (10), § 3,[*] apply for the determination of its factors $\zeta_{i\tau_i}$. And since these formulas involve only the coefficients of u_m, u_{m-1}, which are common to h_{3m} and f_{3m}, it follows that the asymptotic forms of f_{3m} are formally identical with its component forms σ_i when it is degenerate, which proves the statement.*

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[*] To these add Junker's formulas for the determination of linear factors, *Mathem. Annalen*, Bd. 45, S. 39.

Let us investigate the analogous properties of quaternary forms. Let [§ 4]

$$f_{4m}(x, y, z, w) = \sum_{i=0}^{m} u_{m-i}\, w^i.$$

Assume that the ternary form u_m may be decomposed into linear factors

$$u_m = \prod_{k=1}^{m} (x + r_k y + s_k z).$$

Then it is not difficult to show that the necessary and sufficient condition that it be possible to resolve u_{m-1}/u_m into the partial fractions

$$\frac{u_{m-1}}{u_m} = \sum_{k=1}^{m} \frac{\alpha_k}{x + r_k y + s_k z}$$

is the vanishing of each determinant of the matrix

$$M_{m0}^{(m)} = \left|\begin{array}{cccccccc}
a_{m-100} & a_{m-210} & \cdots\cdots a_{0m-10}, & a_{m-201} & a_{m-311} & \cdots\cdots a_{0m-21}, & \cdots\cdots & a_{00m-1} \\
\chi_{00}^{1} & \chi_{10}^{1} & \cdots\cdots \chi_{m-10}^{1}, & \chi_{01}^{1} & \chi_{11}^{1} & \cdots\cdots \chi_{m-21}^{1}, & \cdots\cdots & \chi_{0m-1}^{1} \\
\chi_{00}^{2} & \chi_{10}^{2} & \cdots\cdots \chi_{m-10}^{2}, & \chi_{01}^{2} & \chi_{11}^{2} & \cdots\cdots \chi_{m-21}^{2}, & \cdots\cdots & \chi_{0m-1}^{2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\chi_{00}^{\kappa} & \chi_{10}^{\kappa} & \cdots\cdots \chi_{m-10}^{\kappa}, & \chi_{01}^{\kappa} & \chi_{11}^{\kappa} & \cdots\cdots \chi_{m-21}^{\kappa}, & \cdots\cdots & \chi_{0m-1}^{\kappa} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\chi_{00}^{\lambda} & \chi_{10}^{\lambda} & \cdots\cdots \chi_{m-10}^{\lambda}, & \chi_{01}^{\lambda} & \chi_{11}^{\lambda} & \cdots \chi_{m-21}^{\lambda}, & \cdots\cdots & \chi_{0m-1}^{\lambda} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\chi_{00}^{m} & \chi_{10}^{m} & \cdots\cdots \chi_{m-10}^{m}, & \chi_{01}^{m} & \chi_{11}^{m} & \cdots\cdots \chi_{m-21}^{m}, & \cdots\cdots & \chi_{0m-1}^{m}
\end{array}\right|, \qquad (32)$$

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* An interesting theorem on partial fractions results from the fact that the $\zeta_{i\,\tau_i}$ may be determined both by the partial fraction method and the differential formulas of the preceding sections:

Let

$$\frac{u_{m-1}}{u_m} = \frac{\sum\limits_{i=0}^{m-1} a_{m-1-i\,i}\, x^{m-1-i}\, y^i}{\sum\limits_{j=0}^{m} a_{m-j\,j}\, x^{m-j}\, y^j} = \frac{u_{m-1}}{\prod\limits_{h} (x - r_h y) \prod\limits_{k} (x^2 + \xi_k x y + \eta_k y^2)}.$$

When this is reduced to partial fractions, the numerator of that fraction which has $x^2 + \xi_k xy + \eta_k y^2$ for a denominator may be obtained by differentiation. It is

$$-\frac{\delta \Phi_{-\xi_k}^{(m)}}{\delta' \Phi_{-\xi_k}^{(m)}}\, x - \frac{\delta \Psi_{-\eta_k}^{(m)}}{\delta' \Psi_{-\eta_k}^{(m)}}\, y = v_{k,1},$$

where $\delta' = \dfrac{\partial}{\partial x}$, and δ is the Aronhold operator

$$\delta = a_{m-10} \frac{\partial}{\partial a_{m-11}} + a_{m-21} \frac{\partial}{\partial a_{m-22}} + \cdots\cdots + a_{0\,m-1} \frac{\partial}{\partial a_{0\,m}}.$$

where $\chi^{\lambda}_{\mu\nu}$ is the elementary symmetric function of the $m-1$ groups of variables $P_k(r_k, s_k)$ $[k = 1, 2, \ldots, \lambda-1, \lambda+1, \ldots, m]$, each term of which involves μ r's and ν s's.* If in this matrix the two groups $P_\kappa(r_\kappa, s_\kappa)$, $P_\lambda(r_\lambda, s_\lambda)$ are interchanged, the only effect will be to interchange the κth and λth rows of the matrix. Consequently: *Every determinant of the matrix $M^{(m)}_{m0}$ is an alternating function of the variables r, s.* More generally, the condition that u_{m-1}/u_m be resolvable as follows:

$$\frac{u_{m-1}}{u_m} = \sum_{i=1}^{k} \frac{\alpha_i}{x + r_i y + s_i z} + \sum_{j=1}^{l} \frac{\beta_j x + \gamma_j y + \delta_j z}{x^2 + t_j x y + u_j y^2 + v_j x z + w_j y z + x_j z^2},$$

is the vanishing of a determinate matrix $M^{(m)}_{kl}$. This is a matrix of $\frac{1}{2} m(m-1)$ columns and $\frac{1}{2}(3m-k+2)$ rows, the elements being functions of the coefficients of u_{m-1} and of the factors of u_m.

When u_m can be thus resolved into k linear and $l = \frac{1}{2}(m-k)$ quadratic factors, and $M^{(m)}_{kl} = 0$, then f_{4m} can be thrown into the form

$$f_{4m} = \zeta_{1\tau_1} \zeta_{2\tau_2} \cdots \zeta_{\nu\tau_\nu} + w^2 \mu_{m-2} = 0, \tag{33}$$

where

$$\zeta_{i\tau_i} = v_{i\tau_i} + v_{i\tau_i-1} w + R_{\tau_i-2} w^2,$$

the $v_{i\tau_i}$ being the factors of u_m, while $v_{i\tau_i-1}$ is the numerator of the partial fraction having $v_{i\tau_i}$ for a denominator. As before, R is arbitrary. We now have, placing $R_{\tau_i-2} = 0$, $w = 1$,

$$\prod_{i=1}^{\nu} \zeta_{i\tau_i} = u_m + u_{m-1} + g_{m-2} = h_{4m}(x, y, z, 1).$$

Hence [§ 4, (12) to (21)], to summarize these results: *The necessary and sufficient conditions that a quaternary form $\sum\limits_{i=0}^{m} u_{m-i} w^i$ may possess a set of asymptotic forms of degree ν $(=1, 2)$ are that u_m may be factorable into linear and quadratic factors, and that $M^{(m)}_{kl} = 0$. As the form is altered by small variations of its coefficients, the necessary condition that it degenerate into k linear and l quadratic factors is $M^{(m)}_{kl} = 0$. And in its degenerate form the component factors coincide with the asymptotic forms ζ.*

§ 7. *Summary of Geometrical Results.*

The effect of the differential formulas of § 3, when applied to any form f, is to put in evidence the asymptotic forms of that form. That is, by these formulas we reduce the given form to the type (31) or (33). If the form proves to be degenerate, μ_{m-2} vanishes identically, so that our formulas determine the component factors. The necessary and sufficient conditions that the form f represent a penultimate curve (surface), penultimate to a set of lines and conics (planes and conicoids), are that the coefficients of the terms of the satellite be all infinitesimal. *If by small variations of its coefficients the locus $f = 0$ degenerates into a set of lines and conics (planes and conicoids), these lines and conics (planes and conicoids) are the curves (surfaces) which the asymptotic curves (surfaces) of f, $\zeta_{i r_i} = 0$ $[i = 1, 2, \ldots, \nu]$, approach as limiting curves (surfaces) during that variation.* The functional member of the equation of any penultimate curve (surface) can be expressed as the sum of a product of a set of asymptotic forms and a satellite function having infinitesimal coefficients.

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* Thus if $m = 3$, $\chi^2{}_{11} = r_1 s_3 + r_3 s_1$ (see Junker, *loc. cit.*, p. 4). Also $\chi^{\lambda}{}_{00} = 1$ $[\lambda = 1, 2, \ldots, m]$.

PART II.

ON PENULTIMATE CURVES AND SURFACES.

§ 1. *Conditions for Degeneracy.*

It has been proved that the coefficients of a degenerate ternary form f_{3m} are connected by exactly

$$s = \tfrac{1}{2}(m^2 - \Sigma\, r_i)$$

independent relations, $r_1, r_2, \ldots$ being the respective orders of the component factors. If $r_i = 1$, $s = \frac{1}{2} m(m-1)$. *Hence, since the number of coefficients in the satellite μ_{m-2} is precisely $\frac{1}{2} m(m-1)$, these equated to zero are the necessary and sufficient conditions that f_{3m} degenerate into m straight lines.* We proceed to determine the satellite μ_{m-2} for this case $r_i = 1$, of the forms f_{33}, f_{34}.

Let

$$\begin{aligned} f_{33}(x_1, x_2, 1) &= a_x^3 + b_x^2 + c_x + d, \\ d &= d_0, \\ c_x &= c_0 x_1 + c_1 x_2, \\ b_x^2 &= b_0 x_1^2 + b_1 x_1 x_2 + b_2 x_2^2, \\ a_x^3 &= a_0 x_1^3 + a_1 x_1^2 x_2 + a_2 x_1 x_2^2 + a_3 x_2^3 \\ &= (x_1 - r_1 x_2)(x_1 - r_2 x_2)(x_1 - r_3 x_2) \quad [a_0 = 1]. \end{aligned}$$

By the method of § 5 we get the linear satellite of f_{33},

$$\begin{aligned} D\mu_1 = {} & [S b_{r_1}^2 b_{r_2}^2 (r_3 - r_1)(r_2 - r_3) - c_0 D]\, x_1 + [S b_{r_1}^2 b_{r_2}^2 (r_3 - r_1)(r_2 - r_3)\, r_3 - c_1 D]\, x_2 \\ & - [b_{r_1}^2 b_{r_2}^2 b_{r_3}^2 + d_0 D] = 0 = H_1 x_1 + H_2 x_2 + H_3 \text{ (say)}, \end{aligned} \tag{34}$$

where D is the discriminant of the form a_x^3, and the summation S includes the terms obtained from the given term by the permutations of the cyclic group $S = \{1, (r_1 r_2 r_3), (r_1 r_3 r_2)\}$. The coefficients H_i are symmetric in the roots r_1, r_2, r_3, and so we can express them in terms of the coefficients a_j $[j = 1, 2, 3]$. When this is done, the satellite $D\mu_1$ takes the form given on the following page. *The three independent conditions that f_{33} degenerate into three straight lines are accordingly*

$$H_1 = 0, \quad H_2 = 0, \quad H_3 = 0.$$

H_1	$-H_2$	$-H_3$
$4 a_1 a_2 a_3 b_0^2$	$a_2^2 a_3 b_0^2$	$a_3^2 b_0^3$
$- a_2^3 b_0^2$	$- 3 a_1 a_3^2 b_0^2$	$+ a_1 a_3 b_0 b_1^2$
$- 9 a_3^2 b_0^2$	$+ a_1^2 a_3 b_1^2$	$+ a_1^2 b_0 b_2^2$
$+ 3 a_1 a_3 b_1^2$	$- 3 a_2 a_3 b_1^2$	$- 2 a_2 b_0 b_2^2$
$- a_2^2 b_1^2$	$+ a_1^3 b_2^2$	$- a_2 a_3 b_0^2 b_1$
$+ 3 a_2 b_2^2$	$- 4 a_1 a_2 b_2^2$	$+ a_2^2 b_0^2 b_2$
$- a_1^2 b_2^2$	$+ 9 a_3 b_2^2$	$- 2 a_1 a_3 b_0^2 b_2$
$+ 6 a_1 a_3 b_0 b_2$	$+ 9 a_3^2 b_0 b_1$	$+ 3 a_3 b_0 b_1 b_2$
$- 2 a_2^2 b_0 b_2$	$- a_1 a_2 a_3 b_0 b_1$	$- a_1 a_2 b_0 b_1 b_2$
$+ a_1 a_2^2 b_0 b_1$	$+ 2 a_1^2 a_3 b_0 b_2$	$- a_3 b_1^3$
$+ 3 a_2 a_3 b_0 b_1$	$- 6 a_2 a_3 b_0 b_2$	$- a_1 b_1 b_2^2$
$- 4 a_1^2 a_3 b_0 b_1$	$+ 4 a_2^2 b_1 b_2$	$+ a_2 b_1^2 b_2$
$+ a_1 a_2 b_1 b_2$	$- 3 a_1 a_3 b_1 b_2$	$+ b_2^3$
$- 9 a_3 b_1 b_2$	$- a_1^2 a_2 b_1 b_2$	$+ a_1^2 a_2^2 d_0$
$- a_1^2 a_2^2 c_0$	$+ a_1^2 a_2^2 c_1$	$+ 18 a_1 a_2 a_3 d_0$
$- 18 a_1 a_2 a_3 c_0$	$+ 18 a_1 a_2 a_3 c_1$	$- 4 a_2^3 d_0$
$+ 4 a_2^3 c_0$	$- 4 a_2^3 c_1$	$- 4 a_1^3 a_3 d_0$
$+ 4 a_1^3 a_3 c_0$	$- 4 a_1^3 a_3 c_1$	$- 27 a_3^2 d_0$
$+ 27 a_3^2 c_0$	$- 27 a_3^2 c_1$	

By the same methods we derive the satellite conic of f_{34}. Let

$$f_{34}(x_1, x_2, 1) = a_x^4 + b_x^3 + c_x^2 + d_x + e \quad (a_0 = 1),$$

and

$$a_x^4 = (x_1 - r_1 x_2)(x_1 - r_2 x_2)(x_1 - r_3 x_2)(x_1 - r_4 x_2).$$

Then we get

$$\left.\begin{aligned} D\mu_2 = {} & [S_1 b_{r_1}^3 b_{r_2}^3 (r_1 - r_3)(r_1 - r_4)(r_2 - r_3)(r_2 - r_4)(r_3 - r_4)^2 + c_0 D]\, x_1^2 \\ & + [S_1 b_{r_1}^3 b_{r_2}^3 (r_3 - r_1)(r_1 - r_4)(r_2 - r_3)(r_2 - r_4)(r_3 - r_4)^2 (r_3 + r_4) + c_1 D]\, x_1 x_2 \\ & + [S_1 b_{r_1}^3 b_{r_2}^3 (r_1 - r_3)(r_1 - r_4)(r_2 - r_3)(r_2 - r_4)(r_3 - r_4)^2 r_3 r_4 + c_2 D]\, x_2^2 \\ & + [S_2 b_{r_1}^3 b_{r_2}^3 b_{r_3}^3 (r_1 - r_4)(r_4 - r_2)(r_4 - r_3) + d_0 D]\, x_1 \\ & + [S_2 b_{r_1}^3 b_{r_2}^3 b_{r_3}^3 (r_4 - r_1)(r_4 - r_2)(r_4 - r_3) r_4 + d_1 D]\, x_2 \\ & + [-b_{r_1}^3 b_{r_2}^3 b_{r_3}^3 b_{r_4}^3 + e_0 D] = 0 \\ = {} & I_1 x_1^2 + I_2 x_1 x_2 + I_3 x_2^2 + I_4 x_1 + I_5 x_2 + I_6 = 0. \end{aligned}\right\} \quad (35)$$

Each summation S_1 includes the six terms obtained from the given leading term by operating the three powers of the permutation $(r_1 r_2 r_3)$, and these same three powers preceded by the transposition $(r_1 r_4)$. Each summation S_2 includes all terms obtained from the given leading term by the permutations of the cyclic group $S_2 = \{1, (r_1 r_2 r_3 r_4), (r_1 r_3)\cdot(r_2 r_4), (r_1 r_4 r_3 r_2)\}$. The coefficients I_j are symmetric in the r's. Expressing them in terms of the coefficients of a_x^4 we have the satellite[*] conic. *The six independent conditions that* f_{34} *degenerate into four straight lines are* $I_j = 0$ $[j = 1, 2, 3, 4, 5, 6]$.

It is now evident that the degree (in the coefficients of f_{3m}) of the satellite $D\mu_{m-2}$ of the form f_{3m} is just one greater than the degree of the discriminant of

$$a_x^m = (1, a_1, a_2, \ldots, a_m \between x_1, x_2)^m.$$

Hence the degree[†] *of each of the* $\frac{1}{2} m(m-1)$ *conditions that the general ternary* m*-ic* $f_{3m} = 0$ *degenerate into* m *straight lines is exactly* $2m - 1$.

We derive next the conditions that $f_{33} = 0$ degenerate into a straight line and a conic.

Let
$$a_x^3 = (x_1 - r x_2)\,[x_1^2 + (r + a_1) x_1 x_2 + (r^2 + a_1 r + a_2) x_2^2],$$

where r is real. For the satellite line we get, after multiplying by $(3r^2 + 2a_1 r + a_2)^2$[‡],

$$J_1' x_1 + J_2' x_2 + J_3' = 0,$$

$$J_1' = (b_0 r^2 + b_1 r + b_2)\,[2 b_0 r^2 + (2 a_1 b_0 - b_1) r + (a_2 b_0 - b_2)] \\ \qquad - (c_0 - R_0)(3r^2 + 2a_1 r + a_2)^2,$$

$$J_2' = (b_0 r^2 + b_1 r + b_2)\,[b_1 r^2 + (a_1 b_1 - 2 b_2) r + (a_2 b_1 - a_1 b_2 - a_3 b_0)] \\ \qquad - (c_1 + r R_0)(3r^2 + 2a_1 r + a_2)^2,$$

$$J_3' = (3r^2 + 2a_1 r + a_2)\,[(b_0 r^2 + b_1 r + b_2) R_0 - (3r^2 + 2a_1 r + a_2) d_0].$$

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[*] The satellite of the ternary quartic μ_2 has also been computed. The six coefficients I_j are each about equal in size to the eighth-degree invariant (J_8) of the quintic. These will be published later.

[†] An important application is the result that a pencil of curves $f_{3m} + \lambda \phi_{3m}$ contains at most $2m - 1$ degenerate members consisting of m straight lines. This maximum can be reached. Cf. Hadamard, *loc. cit.*

[‡] This cannot vanish, since a_x has no multiple roots, by hypothesis.

Elimination of R_0 from J_i' $[i = 1, 2, 3]$ gives the two conditions for the degeneracy of f_{33} into a straight line and a conic. There are four cases, corresponding to the four ways in which $J_3' = 0$ can be satisfied.

CASE 1. $b_r^2 = b_0 r^2 + b_1 r + b_2 = 0$, $d_0 = 0$, $R_0 = 0$. The conditions are $c_0 = c_1 = 0$, and the vanishing of the resultant of b_r^2, a_r^3. This is

$$J_1 = \begin{vmatrix} b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \\ a_1 b_0 - b_1 & a_2 b_0 - b_2 & a_3 b_0 & 0 \\ 0 & a_1 b_0 - b_1 & a_2 b_0 - b_2 & a_3 b_0 \end{vmatrix} = 0.$$

CASE 2. $d_0 = 0$, $R_0 = c_0$. Here $b_r = 0$, $c_r = 0$, so that the desired conditions are $J_1 = 0$, $b_0 c_1^2 - b_1 c_1 c_0 + b_2 c_0^2 = 0$.

CASE 3. $d_0 = 0$, $R_0 = 0$, $b_r^2 \neq 0$. The two conditions are the vanishing of the resultants of a_r^3 and the two equations (in r) $J_1' = 0$, $J_2' = 0$, viz.:

$$J_2 = \begin{vmatrix} 1 & a_1 & a_2 & a_3 & 0 \\ 0 & 1 & a_1 & a_2 & a_3 \\ \begin{array}{l} a_1 b_0 b_1 \\ - b_1^2 \\ + b_0 b_2 \\ - a_2 b_0^2 \\ + 3 a_2 c_0 \\ - a_1^2 c_0 \end{array} & \begin{array}{l} 2 a_1 b_0 b_2 \\ - 2 a_3 b_0^2 \\ - 2 b_1 b_2 \\ + 9 a_3 c_0 \\ - a_1 a_2 c_0 \end{array} & \begin{array}{l} a_2 b_0 b_2 \\ - a_3 b_0 b_1 \\ - b_2^2 \\ + 3 a_1 a_3 c_0 \\ - a_2^2 c_0 \end{array} & 0 & 0 \\ 0 & \begin{array}{l} a_1 b_0 b_1 \\ - b_1^2 \\ + b_0 b_2 \\ - a_2 b_0^2 \\ + 3 a_2 c_0 \\ - a_1^2 c_0 \end{array} & \begin{array}{l} 2 a_1 b_0 b_2 \\ - 2 a_3 b_0^2 \\ - 2 b_1 b_2 \\ + 9 a_3 c_0 \\ - a_1 a_2 c_0 \end{array} & \begin{array}{l} a_2 b_0 b_2 \\ - a_3 b_0 b_1 \\ - b_2^2 \\ + 3 a_1 a_3 c_0 \\ - a_2^2 c_0 \end{array} & 0 \\ 0 & 0 & \begin{array}{l} a_1 b_0 b_1 \\ - b_1^2 \\ + b_0 b_2 \\ - a_2 b_0^2 \\ + 3 a_2 c_0 \\ - a_1^2 c_0 \end{array} & \begin{array}{l} 2 a_1 b_0 b_2 \\ - 2 a_3 b_0^2 \\ - 2 b_1 b_2 \\ + 9 a_3 c_0 \\ - a_1 a_2 c_0 \end{array} & \begin{array}{l} a_2 b_0 b_2 \\ - a_3 b_0 b_1 \\ - b_2^2 \\ + 3 a_1 a_3 c_0 \\ - a_2^2 c_0 \end{array} \end{vmatrix} = 0,$$

$$J_3 = \begin{vmatrix} 1 & a_1 & a_2 & a_3 & 0 \\ 0 & 1 & a_1 & a_2 & a_3 \\ \begin{array}{r} a_1 b_0 b_2 \\ - a_3 b_0^2 \\ - b_1 b_2 \\ - a_1^2 c_1 \\ + 3 a_2 c_1 \end{array} & \begin{array}{r} 2 a_2 b_0 b_2 \\ - 2 a_3 b_0 b_1 \\ - 2 b_2^2 \\ + 9 a_3 c_1 \\ - a_1 a_2 c_1 \end{array} & \begin{array}{r} a_2 b_1 b_2 \\ - a_1 b_2^2 \\ - a_2^2 c_1 \\ + 3 a_1 a_3 c_1 \\ + a_3 b_0 b_2 \\ - a_3 b_1^2 \end{array} & 0 & 0 \\ 0 & \begin{array}{r} a_1 b_0 b_2 \\ - a_3 b_0^2 \\ - b_1 b_2 \\ - a_1^2 c_1 \\ + 3 a_2 c_1 \end{array} & \begin{array}{r} 2 a_2 b_0 b_2 \\ - 2 a_3 b_0 b_1 \\ - 2 b_2^2 \\ + 9 a_3 c_1 \\ - a_1 a_2 c_1 \end{array} & \begin{array}{r} a_2 b_1 b_2 \\ - a_1 b_2^2 \\ - a_2^2 c_1 \\ + 3 a_1 a_3 c_1 \\ + a_3 b_0 b_2 \\ - a_3 b_1^2 \end{array} & 0 \\ 0 & 0 & \begin{array}{r} a_1 b_0 b_2 \\ - a_3 b_0^2 \\ - b_1 b_2 \\ - a_1^2 c_1 \\ + 3 a_2 c_1 \end{array} & \begin{array}{r} 2 a_2 b_0 b_2 \\ - 2 a_3 b_1 b_2 \\ - 2 b_2^2 \\ + 9 a_3 c_1 \\ - a_1 a_2 c_1 \end{array} & \begin{array}{r} a_2 b_1 b_2 \\ - a_1 b_2^2 \\ - a_2^2 c_1 \\ + 3 a_1 a_3 c_1 \\ + a_3 b_0 b_2 \\ - a_3 b_1^2 \end{array} \end{vmatrix} = 0.$$

CASE 4. $d_{0j}^{3} \neq 0$. By a simple real translation of the y-axis this case is reduced back to one of the first three cases.

We consider next the algebraical conditions for degenerate surfaces. Let

$$f_{4m} = a_x^m + b_x^{m-1} + \dots\dots + k_x + l,$$

where

$$a_x^m = \sum_{j=0}^{m} (a_{0\,m-j}\, x_1^{m-j} + a_{1\,m-j}\, x_1^{m-j-1} x_2 + \dots\dots + a_{m-j\,m-j}\, x_2^{m-j})\, x_3^j \quad (a_{0m} = 1),$$

$$b_x^{m-1} = \sum_{i=0}^{m-1} (b_{0\,m-i-1}\, x_1^{m-i-1} + b_{1\,m-i-1}\, x_1^{m-i-2} x_2 + \dots\dots + b_{m-i-1\,m-i-1}\, x_2^{m-i-1})\, x_3^i,$$

. . . .

Suppose that f_{4m} is degenerate, the component forms being $\varkappa$ linear and $\lambda = \frac{1}{2}(m - \varkappa)$ quadratic forms. Then the n_1 conditions for the decomposition of the form a_x^m into $\varkappa$ linear and λ quadratic ternary forms must be fulfilled, each determinant of the matrix $M_{\kappa\lambda}^{(m)}$ vanishes, and each separate coefficient of the satellite μ_{m-2} in

$$f_{4m} = \Pi\, \zeta_{i\tau_i} + \mu_{m-2}$$

vanishes. The number n_2 of independent determinants of the matrix $M_{\kappa\lambda}^{(m)}$ is $\frac{1}{2}(m^2 - 2m + \varkappa)$; and the number of coefficients of the quaternary form μ_{m-2} is $n_3 = \frac{1}{6} m (m-1)(m+1)$. Hence, the number of independent conditions *in toto* obtainable from these three classes of conditions is

$$\begin{aligned} n_1 + n_2 + n_3 &\equiv \tfrac{1}{2}(m^2 - 2m + \varkappa) + \tfrac{1}{2}(m^2 - 2m + \varkappa) + \tfrac{1}{6} m (m-1)(m+1) \\ &= \tfrac{1}{6} m (m-1)(m+7) - (m - \varkappa). \end{aligned}$$

But this is the well-known total number of conditions in order that f_{4m} degenerate into $\varkappa$ linear and λ quadratic factors. Hence:

The totality of independent conditions that the general m-ic surface $f_{4m} = 0$ degenerate into $\varkappa$ planes and λ conicoids consists in the conditions that a_x^m decompose into $\varkappa$ linear and λ quadratic ternary factors, the vanishing of any complete set of independent determinants of the matrix $M_{\kappa\lambda}^{(m)}$, and the vanishing of each coefficient of the satellite μ_{m-2} of the form f_{4m}.

The explicit forms of these conditions will now be derived for the two cases $m = 2$, $\varkappa = 2$, $\lambda = 0$; and $m = 3$, $\varkappa = 3$, $\lambda = 0$. Let

$$a_x^2 = (x_1 + r_1 x_2 + s_1 x_3)(x_1 + r_2 x_2 + s_2 x_3).$$

Then we have

$$\begin{aligned} &a_{12} = r_1 + r_2, \quad a_{22} = r_1 r_2, \quad a_{01} = s_1 + s_2, \quad a_{11} = r_1 s_2 + r_2 s_1, \quad a_{00} = s_1 s_2, \\ &M_{20}^{(2)} = b_{01}(r_2 s_1 - r_1 s_2) + b_{11}(s_2 - s_1) + b_{00}(r_1 - r_2), \\ &\mu_0 = b_{01}^2 r_1 r_2 - b_{01} b_{11}(r_1 + r_2) + c_{00}(r_1 - r_2)^2. \end{aligned}$$

Hence, the three conditions necessary for the degeneracy of the general conicoid $f_{42} = 0$ into two planes are, say, $\varkappa_i = 0$ $(i = 1, 2, 3)$, where

$$\begin{aligned} \varkappa_1 &= 4 a_{00} a_{22} + a_{01} a_{11} a_{12} - a_{11}^2 - a_{22} a_{01}^2 - a_{00} a_{12}^2, \\ \varkappa_2 &= a_{22} b_{01}^2 + b_{11}^2 - a_{12} b_{01} b_{11} + a_{12}^2 c_{00} - 4 a_{22} c_{00}, \\ \varkappa_3 &= b_{01} \sqrt{a_{11}^2 - 4 a_{00} a_{22}} + b_{11} \sqrt{a_{01}^2 - 4 a_{00}} + b_{00} \sqrt{a_{12}^2 - 4 a_{22}}. \end{aligned}$$

Next let

$$a_x^3 = (x_1 + r_1 x_2 + s_1 x_3)(x_1 + r_2 x_2 + s_2 x_3)(x_1 + r_3 x_2 + s_3 x_3).$$

Then

$$\begin{aligned} &a_{13} = \Sigma\, r_1, \quad a_{23} = \Sigma\, r_1 r_2, \quad a_{33} = r_1 r_2 r_3, \quad a_{02} = \Sigma\, s_1, \\ &a_{12} = \Sigma\, r_1 s_2, \quad a_{22} = \Sigma\, r_1 r_2 s_3, \quad a_{01} = \Sigma\, s_1 s_2, \quad a_{11} = \Sigma\, r_1 s_2 s_3, \quad a_{00} = s_1 s_2 s_3. \end{aligned}$$

Let D_r, D_s represent the discriminants of $r^3 + a_{13} r^2 + a_{23} r + a_{33}$ and $s^3 + a_{02} s^2 + a_{01} s + a_{00}$ respectively. Also $\beta_r^2 = b_{02} r^2 - b_{12} r + b_{22}$, $\gamma_s^2 = b_{02} s^2 - b_{01} s + b_{00}$. Then we get for the satellite μ_1 of f_{43}

$$\begin{aligned} -\mu_1 = {} & \frac{1}{D_r}\left[S\beta_{r_1}^2\beta_{r_2}^2(r_2-r_3)(r_3-r_1)-c_{01}D_r\right]x_1 \\ & + \frac{1}{D_r}\left[S\beta_{r_1}^2\beta_{r_2}^2(r_2-r_3)(r_3-r_1)r_3-c_{11}D_r\right]x_2 \\ & + \frac{1}{D_s}\left[S\gamma_{s_1}^2\gamma_{s_2}^2(s_2-s_3)(s_3-s_1)s_3-c_{00}D_s\right]x_3 - \frac{1}{D_r}\left[\beta_{r_1}^2\beta_{r_2}^2\beta_{r_3}^2+d_{00}D_r\right] = 0, \end{aligned}$$

where the summation S includes the three terms obtained from the given leading term by the substitutions of the cyclic group $S = \{1, (123), (132)\}$. Comparing this with equation (34), we may at once state the following conclusions: Let the three conditions that a_x^3 break into three linear ternary factors be L_1, L_2, L_3, the four conditions obtained by equating the above coefficients of the satellite μ_1 to zero be $L_i\,(i = 4, 5, 6, 7)$, and the three independent determinants of the matrix $M_{30}^{(3)}$ be L_8, L_9, L_{10}. Then L_i is given at once from $H_i\,(i = 1, 2, 3)$ by operating upon the latter the substitution $\begin{pmatrix} a_j & b_k & c_l & d_0 \\ a_{j3} & a_{k2} & a_{l1} & a_{00} \end{pmatrix}$. The first two and the last coefficient of μ_1 are obtainable from $H_i\,(i = 1, 2, 3)$ by the substitution $\begin{pmatrix} a_j & b_k & c_l & d_0 \\ a_{j3} & b_{k2} & c_{l1} & d_{00} \end{pmatrix}$, and the coefficient of x_3 is given by operating upon H_2 the substitution $\begin{pmatrix} a_j & b_k & c_l \\ a_{03-j} & b_{02-k} & c_{01-l} \end{pmatrix}$. Thus $L_i\,(i = 1, \ldots, 7)$ are known explicitly. It remains to determine L_8, L_9, L_{10}. The matrix belonging to this case is

$$M_{30}^{(3)} = \begin{vmatrix} b_{02} & b_{12} & b_{22} & b_{01} & b_{00} & b_{11} \\ 1 & r_2 + r_3 & r_2 r_3 & s_2 + s_3 & s_2 s_3 & r_2 s_3 + r_3 s_2 \\ 1 & r_1 + r_3 & r_1 r_3 & s_1 + s_3 & s_1 s_3 & r_1 s_3 + r_3 s_1 \\ 1 & r_1 + r_2 & r_1 r_2 & s_1 + s_2 & s_1 s_2 & r_1 s_2 + r_2 s_1 \end{vmatrix} = 0.$$

Hence, we may select for our three matrix conditions

$$|b_{02}\,\chi_{10}^{1}\,\chi_{20}^{2}\,\chi_{01}^{3}| = 0, \quad |b_{02}\,\chi_{10}^{1}\,\chi_{01}^{2}\,\chi_{02}^{3}| = 0, \quad |b_{02}\,\chi_{10}^{1}\,\chi_{01}^{2}\,\chi_{11}^{3}| = 0,$$

as $L_i\,(i = 8, 9, 10)$ respectively. Each determinant is an alternating function of the groups P_1, P_2, P_3 and hence can be expressed in terms* of square roots of rational functions of the coefficients a. The explicit forms of these three conditions are as follows:*

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* These computations are carried out with the aid of tables of symmetric functions of two groups of variables; namely, Junker: "Die symmetrischen Functionen der gemeinschaftlichen Variabelnpaare ternärer Formen: Tafeln der symmetrischen Functionen vom Gewicht 1 bis 6," *Wiener Denkschriften*, 1897.

$$\begin{aligned}
L_8 = {}& b_{01}\,(a_{13}^2\,a_{23}^2 + 18\,a_{13}\,a_{23}\,a_{33} - 4\,a_{23}^3 - 4\,a_{13}^3\,a_{33} - 27\,a_{33}^2)^{\frac{1}{3}} + b_{22}\,(12\,a_{23}\,a_{01} - 3\,a_{12}^2 \\
& - 4\,a_{02}^2\,a_{23} + 4\,a_{13}\,a_{02}\,a_{12} - 4\,a_{13}^2\,a_{01})^{\frac{1}{3}} + 15^{-\frac{1}{3}}b_{12}\,(-14\,a_{13}^4\,a_{01} + 14\,a_{13}^3\,a_{02}\,a_{12} - 14\,a_{13}^2\,a_{02}^2\,a_{23} \\
& + 52\,a_{13}^2\,a_{23}\,a_{01} - 14\,a_{13}^2\,a_{12}^2 + 11\,a_{02}^2\,a_{23}^2 + 4\,a_{13}\,a_{02}\,a_{23}\,a_{12} + 14\,a_{13}^3\,a_{11} - 28\,a_{13}^2\,a_{02}\,a_{22} \\
& - 18\,a_{13}\,a_{02}^2\,a_{33} - 44\,a_{23}^2\,a_{01} - 4\,a_{23}\,a_{12}^2 + 12\,a_{02}\,a_{12}\,a_{33} + 42\,a_{13}\,a_{01}\,a_{33} + 22\,a_{02}\,a_{23}\,a_{22} \\
& + 42\,a_{13}\,a_{12}\,a_{22} + 52\,a_{13}\,a_{23}\,a_{11} - 33\,a_{22}^2 - 36\,a_{33}\,a_{11})^{\frac{1}{3}} + b_{02}\,(a_{13}^2\,a_{22}^2 - 2\,a_{13}\,a_{02}\,a_{33}\,a_{22} \\
& + 3\,a_{02}^2\,a_{33}^2 - 2\,a_{13}^3\,a_{01}\,a_{33} + 2\,a_{13}^2\,a_{02}\,a_{12}\,a_{33} - 2\,a_{13}\,a_{02}^2\,a_{23}\,a_{33} - 2\,a_{13}\,a_{12}^2\,a_{33} + 8\,a_{13}\,a_{23}\,a_{01}\,a_{33} \\
& - 2\,a_{13}^2\,a_{11}\,a_{33} + 2\,a_{23}\,a_{11}\,a_{33} + 2\,a_{12}\,a_{22}\,a_{33} - 14\,a_{01}\,a_{33}^2 - 4\,\Sigma\, r_1^3\,r_2^3\,s_3^2)^{\frac{1}{3}} + b_{02}\,(a_{23}^2\,a_{12}^2 + a_{13}^2\,a_{22}^2 \\
& - a_{02}^2\,a_{33}^2 - 2\,a_{13}\,a_{23}\,a_{12}\,a_{22} - 2\,a_{02}\,a_{23}\,a_{12}\,a_{33} - 2\,a_{01}\,a_{33}^2 + 2\,a_{13}\,a_{02}\,a_{33}\,a_{22} - 2\,a_{13}^3\,a_{01}\,a_{33} \\
& + 2\,a_{13}^2\,a_{02}\,a_{12}\,a_{33} - 2\,a_{13}\,a_{02}^2\,a_{23}\,a_{33} - 2\,a_{13}\,a_{12}^2\,a_{33} + 8\,a_{13}\,a_{23}\,a_{01}\,a_{33} - 2\,a_{13}^2\,a_{11}\,a_{33} + 2\,a_{23}\,a_{11}\,a_{33} \\
& + 2\,a_{12}\,a_{22}\,a_{33} - 4\,\Sigma\, r_1^3\,r_2^3\,s_1\,s_2)^{\frac{1}{3}} = 0.
\end{aligned}$$

The determinant L_9 is given by operating upon L_8 the substitution $\begin{pmatrix} a_{j\,3-i} & b_{01} & b_{02} & b_{12} & b_{22} \\ a_{i\,3-j} & b_{12} & b_{02} & b_{01} & b_{00} \end{pmatrix}$.

$$\begin{aligned}
L_{10} = {}& b_{11}(12a_{13}a_{01} - 3\,a_{12}^2 - 4\,a_{02}^2\,a_{23} + 4\,a_{13}\,a_{02}\,a_{12} - 4\,a_{13}^2\,a_{01})^{\frac{1}{3}} + 15^{-\frac{1}{3}}\,b_{01}(-64\,a_{13}^4\,a_{01} \\
& + 64\,a_{13}^3\,a_{02}\,a_{12} - 64\,a_{13}^2\,a_{02}^2\,a_{23} + 252\,a_{13}^2\,a_{23}\,a_{01} - 49\,a_{13}^2\,a_{12}^2 - 26\,a_{13}\,a_{02}\,a_{23}\,a_{12} + 11\,a_{02}^2\,a_{23}^2 \\
& + 4\,a_{13}^3\,a_{11} - 68\,a_{13}^2\,a_{02}\,a_{22} + 132\,a_{13}\,a_{02}^2\,a_{33} - 44\,a_{23}^2\,a_{01} - 4\,a_{23}\,a_{12}^2 + 12\,a_{02}\,a_{12}\,a_{33} \\
& - 408\,a_{13}\,a_{01}\,a_{33} + 22\,a_{02}\,a_{23}\,a_{22} + 102\,a_{13}\,a_{12}\,a_{22} - 33\,a_{22}^2 + 82\,a_{13}\,a_{23}\,a_{11} - 36\,a_{33}\,a_{11})^{\frac{1}{3}} \\
& + 15^{-\frac{1}{3}}\,b_{12}\,(-14\,a_{02}^4\,a_{23} + 14\,a_{02}^3\,a_{13}\,a_{12} - 14\,a_{02}^2\,a_{13}^2\,a_{01} + 52\,a_{02}^2\,a_{01}\,a_{23} - 14\,a_{02}^2\,a_{12}^2 \\
& + 4\,a_{02}\,a_{13}\,a_{01}\,a_{12} + 11\,a_{13}^2\,a_{01}^2 + 14\,a_{02}^3\,a_{22} - 28\,a_{02}^2\,a_{13}\,a_{11} - 18\,a_{02}\,a_{13}^2\,a_{00} - 44\,a_{01}^2\,a_{23} \\
& - 4\,a_{01}\,a_{12}^2 + 12\,a_{13}\,a_{12}\,a_{00} + 42\,a_{02}\,a_{23}\,a_{00} + 22\,a_{13}\,a_{01}\,a_{11} + 42\,a_{02}\,a_{12}\,a_{11} - 33\,a_{11}^2 \\
& + 52\,a_{02}\,a_{01}\,a_{22} - 36\,a_{00}\,a_{22})^{\frac{1}{3}} + 15^{-\frac{1}{3}}\,b_{12}\,(-64\,a_{02}^4\,a_{23} + 64\,a_{02}^3\,a_{13}\,a_{12} - 64\,a_{02}^2\,a_{13}^2\,a_{01} \\
& + 252\,a_{02}^2\,a_{01}\,a_{23} - 49\,a_{02}^2\,a_{12}^2 - 26\,a_{02}\,a_{13}\,a_{01}\,a_{12} + 11\,a_{13}^2\,a_{01}^2 + 4\,a_{02}^3\,a_{22} - 68\,a_{02}^2\,a_{13}\,a_{11} \\
& + 132\,a_{02}\,a_{13}^2\,a_{00} - 44\,a_{01}^2\,a_{23} - 4\,a_{01}\,a_{12}^2 + 12\,a_{13}\,a_{12}\,a_{00} - 408\,a_{02}\,a_{23}\,a_{00} + 22\,a_{13}\,a_{01}\,a_{11} \\
& + 102\,a_{02}a_{12}a_{11} - 33\,a_{11}^2 + 82\,a_{02}a_{01}a_{22} - 36\,a_{00}\,a_{22})^{\frac{1}{3}} + 15^{-\frac{1}{3}}\,b_{01}(-14a_{13}^4\,a_{01} + 14\,a_{13}^3\,a_{02}\,a_{12} \\
& - 14\,a_{13}^2\,a_{02}^2\,a_{23} + 52\,a_{13}^2\,a_{23}\,a_{01} - 14\,a_{13}^2\,a_{12}^2 + 4\,a_{13}\,a_{02}\,a_{23}\,a_{12} + 11\,a_{02}^2\,a_{23}^2 + 14\,a_{13}^3\,a_{11} \\
& - 28\,a_{13}^2\,a_{02}\,a_{22} - 18\,a_{13}\,a_{02}^2\,a_{33} - 44\,a_{23}^2\,a_{01} - 4\,a_{23}\,a_{12}^2 + 12\,a_{02}\,a_{12}\,a_{33} + 42\,a_{13}\,a_{01}\,a_{33} \\
& + 22\,a_{02}\,a_{23}\,a_{22} + 42\,a_{13}\,a_{12}\,a_{22} - 33\,a_{22}^2 + 52\,a_{13}\,a_{23}\,a_{11} - 36\,a_{33}\,a_{11})^{\frac{1}{3}} + 3^{-1}\,b_{02}\,(-36\,\Sigma\, r_1^4\,s_2^2\,s_3^2 \\
& - 36\,\Sigma\, r_1^2\,r_2^2\,s_3^4 - 36\,\Sigma\, r_1^2\,r_2^2\,s_1^2\,s_2^2 + (-4\,a_{13}^2\,a_{01} + 4\,a_{13}\,a_{02}\,a_{12} - 4\,a_{02}^2\,a_{23} - a_{12}^2 + 10\,a_{23}\,a_{01} \\
& - 2\,a_{13}\,a_{11} - 2\,a_{02}\,a_{22})^2)^{\frac{1}{3}} + \tfrac{2}{3}\,b_{02}\,((-a_{13}^2\,a_{01} + a_{13}\,a_{02}\,a_{12} - a_{02}^2\,a_{23} - a_{12}^2 + 4\,a_{23}\,a_{01} + a_{13}\,a_{11} \\
& + a_{02}\,a_{22})^2 - 36\,(a_{00}\,a_{22}\,a_{23} + a_{00}\,a_{02}\,a_{13}\,a_{33} - 3\,a_{00}\,a_{12}\,a_{33} + a_{01}\,a_{11}\,a_{33}))^{\frac{1}{3}} = 0.
\end{aligned}$$

§ 2. *Degenerate and Penultimate Loci.*

The coefficients of the satellite function μ_{m-2} in

$$f = \Pi\,\zeta_{i\,r_i} + \mu_{m-2} = 0$$

are rational functions of the coefficients of f, and linear functions of the

$\Sigma \frac{1}{2} \tau_i (\tau_i - 1)$ arbitrary coefficients R of $\zeta_{i\tau_i}$. The set of equations obtained by equating μ_{m-2} identically to zero can sometimes be satisfied in only a finite number of ways, in which event the curve (surface) $f = 0$ has only a finite number of degenerate forms.

For example, the cubic

$$(x - \alpha y)(y - \beta z)(z - \gamma x) + (x - \alpha z)(y - \beta x)(z - \gamma y) = 0$$

reduced to the form (31) is

$$\begin{aligned} &[\gamma(\alpha-1)x + \alpha(\gamma-1)][\gamma(\beta-1)y + \beta(\gamma-1)][\gamma(\beta-1)x + \gamma(\alpha-1)y + R] \\ &\quad - \left[R - \gamma\frac{(\alpha-1)(\beta-1)}{\gamma-1}\right][\beta\gamma(\alpha-1)(\gamma-1)x + \alpha\gamma(\beta-1)(\gamma-1)y] \\ &\quad - \alpha\beta(\gamma-1)^2[\gamma(\beta-1)x + \gamma(\alpha-1)y + R] = 0. \end{aligned}$$

Hence, there are two degenerate forms of the curve corresponding to the two cases

(a) $R\left(= 2(1 - \alpha\beta\gamma) - \frac{\alpha(\beta-1)(\gamma-1)}{\alpha-1} - \frac{\beta(\alpha-1)(\gamma-1)}{\beta-1}\right) = \frac{\gamma(\alpha-1)(\beta-1)}{\gamma-1}$,

(b) $R = 0, \ \alpha(\beta-1)^2 = \beta(\alpha-1)^2 = \frac{\alpha\beta}{\gamma}(\gamma-1)^2$.

They are respectively *

$$\left(\frac{x}{\alpha-1} + \frac{y}{\beta-1} + \frac{1}{\gamma-1}\right)\left(\frac{\beta}{\beta-1}x + \frac{\alpha}{\alpha-1}y + \frac{\gamma}{\gamma-1}xy\right) = 0,$$

$$\left(\frac{\gamma}{\gamma-1}x + \frac{\alpha}{\alpha-1}\right)\left(\frac{\gamma}{\gamma-1}y + \frac{\beta}{\beta-1}\right)\left(\frac{\beta}{\beta-1}x + \frac{\alpha}{\alpha-1}y\right) = 0.$$

When the general cubic curve $f_{3m} = 0$ is treated in this manner, we find six degenerate forms, three consisting of triads of lines, and three of a line and a conic.

It is to be observed that each degenerate (ultimate) form of a curve or suface has corresponding to it a large number of penultimate forms. That is, a locus can pass over into any degenerate form in a number of different ways; and it is possible to enumerate all the penultimate curves which correspond to any determined degenerate form of a curve. Thus the penultimate quartic curve

$$\zeta_1 \zeta_2 \zeta_3 \zeta_4 + \varepsilon^2 \mu_2 = 0 \quad (\varepsilon \doteq 0)$$

crosses its asymptotes ζ_i at those points in which the latter are cut by the satellite conic $\mu_2 = 0$, and in no other points. Let this conic be chosen such that it encloses all six vertices of the quadrilateral formed by the asymptotes. Then

$$\zeta_1 = -\varepsilon^2 \mu_2 / \zeta_2 \zeta_3 \zeta_4 .$$

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* Cf. Taylor, *loc. cit.*

Hence, for points of the curve approximating to the segment $\overline{ab}$, ζ_2, ζ_3, ζ_4 are negative, μ_2 positive; so that the curve is on the side of $\overline{ab}$ opposite from the origin. In this manner the curve may be traced completely. Its shape in the vicinity of the vertex c should be noted particularly.

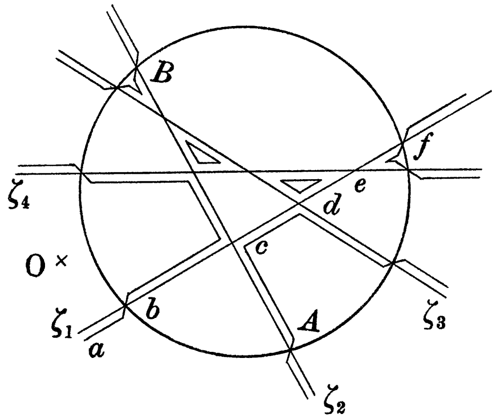

FIG. 1.

Now let the satellite conic vary in position, its perimeter approaching c and crossing it, while the relative positions of the conic, the origin and the other five vertices remain unaltered. When $\mu_2=0$ crosses c, the latter will be a double point of the penultimate curve; and when c becomes exterior to the curve $\mu_2=0$, the shape of the quartic near c is as shown in Fig. 2. Thus, by variation of the

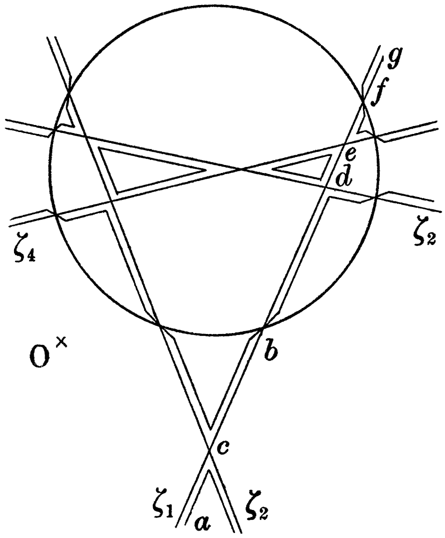

FIG. 2.

position of the satellite conic three distinct cases are produced at each vertex, making 3^6 ways of drawing curves penultimate to four straight lines. Not all of these types are distinct as types. Also Plücker's laws must be satisfied. It is found, for example, that the number of distinct types of penultimate quartics with four rectilinear asymptotes, and having no point singularities at infinity, is 343.

Consider also quartics having two asymptotic ellipses intersecting in four real points

$$\zeta_{12}\zeta_{22} - \epsilon^2\mu_2 = 0 \quad (\epsilon \doteq 0).$$

If the conic $\mu_2 = 0$ encloses all of these points, the quartic is the four-ovaled type (Fig. 3). If the satellite encloses none of the points, the penultimate curve is the type having one nested oval. One of these types may be made to pass into the other by continuous variation, by small variations of the coefficients of the satellite $\mu_2 = 0$.

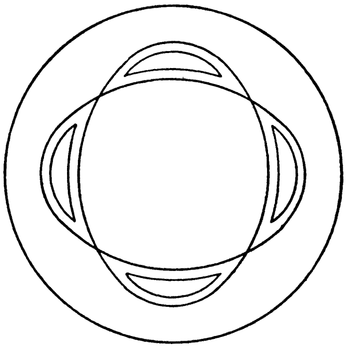

FIG. 3.

Analogously, along a line of intersection of a pair of asymptotic planes of any penultimate surface there are three possible cases of the relative positions of surface and planes. Consider the penultimate cubic surfaces having three asymptotic planes L, M, N intersecting in the point C and lines $\overline{CA}$, $\overline{CB}$, $\overline{CD}$. Let the satellite plane intersect $\overline{CA}$, $\overline{CB}$, $\overline{CD}$ respectively in the points A, B, D Let the origin O be on the side of the satellite plane $\overline{ABD}$ opposite from C and inside the trihedral angle $\overline{ABD} - C$. Then along the line $\overline{CA}$ from infinity up to A the cubic surface is in the dihedral angle which contains the origin, and its vertical. But from A to C the surface lies in the other pair of vertical dihedral angles. Similarly the surface may be traced along the other asymptotic planes. It hangs together at the points A, B, D and at no other points. By shifting the satellite plane into all typical positions with reference to the origin and the asymptotic planes, we get all the possible types of cubic surface penultimate to three planes.

§ 3. *Penultimate Class Loci.*

The case where, in f_{3m}, a_x^m has equal roots, excluded in most of the previous discussion, is best treated from the standpoint of the line equation. This subject we may introduce by the following quotation from Cayley:[*] "The system of tangents from an arbitrary point to the penultimate curve reduces (when the curve approaches degeneracy) to 1) the tangents to the several component curves; 2) the lines through the singular points of these same curves respectively; 3) the lines through the points of intersection of each two of the component curves; these points, each reckoned the proper number of times, are called 'fixed summits'; 4) the lines from the arbitrary point to certain determinate points called 'free summits' on the several component curves . . . . Thus a degenerate form of the n-ic curve . . . . may be regarded as consisting of the component curves, each its proper number of times, and the foregoing points called summits, and is consequently only inadequately represented by the ultimate form of the equation ($\zeta_{1\tau_1}\zeta_{2\tau_2}\dots\zeta_{\nu\tau_\nu}=0$)."

The line equation of $f_{3m}=0$, say $\phi_{3n}=0$, can, by the methods of Part I, be thrown into the form

$$\phi_{3n}=\eta_{1\rho_1}\eta_{2\rho_2}\dots\eta_{\mu\rho_\mu}+\psi_{n-2}=0.$$

Keeping the restriction $\tau_i=1$ or 2, we have $\rho_j=1$ or 2. Then the $\eta_{j\rho_j}=0$ ($\rho_j=2$) are the line equations of the asymptotic conics of the penultimate $f_{3m}=0$. But the linear forms η can represent only *points on* the linear constituents of the penultimate $f_{3m}=0$. These points are in fact the before-mentioned "summits" of the curve. Thus the cubic

$$\begin{aligned}f_{33}=x_1^3+x_2^3+p\,x_1+v\,x_2&=(x_1+x_2)(x_1^2-x_1x_2+x_2^2+R)\\&\quad+(p-R)x_1+(v-R)x_2=\zeta_{11}\zeta_{22}+\mu_1=0\end{aligned}$$

degenerates when $R=p=v$ into line and ellipse $\zeta_{11}\zeta_{22}=0$. The class equation of $\zeta_{11}\zeta_{22}=0$ is

$$\begin{aligned}\phi_{36}=\eta_{11}^2\eta_{21}^2\eta_{32}=\left(u_1-u_2+\sqrt{\frac{3}{-v}}u_3\right)^2\left(u_1-u_2-\sqrt{\frac{3}{-v}}u_3\right)^2&\\(4\,v\,u_1^2+4\,v\,u_2^2+4\,v\,u_1u_2+3\,u_3^2)=0,&\end{aligned}$$

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[*] Cayley, *loc. cit.*

and $\eta_{32}=0$ is the class equation of $\zeta_{22}=0$, whereas $\eta_{11}^2=0$, $\eta_{21}^2=0$ are the points of intersection of $\zeta_{11}=0$ and $\zeta_{22}=0$, the fixed summits, each counting double. Conditions similar to the above hold when $\tau_i > 2$.

It is well-known that the class equation of a curve $f=0$ can be reduced to the form*

$$\rho_1\rho_2\cdots\cdot\rho_n+\omega\omega'\psi'_{n-2}=0,$$

where the ρ_i are the foci of the curve and ω, ω' are the circular points at infinity. When the non-linear factors of the degenerate ϕ_{3n} are each put in this form, we have

$$\phi_{3n}=\eta_{11}\eta_{21}\cdots\cdot\eta_{\mu 1}\rho_1\rho_2\cdots\cdot\rho_1'\rho_2'\cdots\cdot+\omega\omega'\psi''_{n-2}=0.$$

Hence, *when by small variations of its coefficients the class locus $\phi_{3n}=0$ degenerates, it degenerates into those curves which its asymptotes approach as limiting curves, together with those points which its foci approach as limiting points, during the variation, the foci of the non-linear constituents being excepted.*

Whenever a penultimate curve $f_{3m}=0$ possesses a proper class equation $\phi_{3n}=0$, the latter's linear constituents represent only *fixed* summits (and double points). The function ϕ_{3n} vanishes identically only in case the penultimate f_{3m} has a multiple asymptote. Therefore *free summits can exist only upon penultimate curves having multiple asymptotes.* Since in this case the class equation is evanescent and the degree equation gives no evidence of the positions of the summits, the analytical determination of the free summits depends upon special methods. Cayley and Zeuthen have published tentative methods of finding the number of these summits. By using symbolical methods, the details of which are too long for the scope of this paper, I have developed a method of finding the multiplicities of the fixed summits. If n_i $(i=1, 2, \ldots .)$ represents the number of fixed summits of species 1, 2, ... and m_i the multiplicity of n_i, also if q is the number of free summits and the classes of the non-linear (and non-singular, say) constituents of ϕ_{3n} are respectively r_1, r_2, ..., we have of course a relation

$$q=n-\Sigma\, m_i n_i-\Sigma\, r_j.$$

Zeuthen has in certain special problems determined the equation of a curve which intersects the multiple asymptotes in the free summits, thus determining geometrically the positions of the latter. It should be noted that, in general, *the intersections of the satellite $\mu_{m-2}=0$ with the multiple asymptotes of the penultimate curve $f_{3m}=0$ are all free summits of the latter curve.*

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* C. A. Scott, Modern Analytic Geometry, § 270.

UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PA.